Part 1. Analysis of the Theodorus Spiral During One Planck Step

(a Simulation Hypothesis model)

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Reference programme: `http://codingthecosmos.com/Theodorus-Omega-base15.py`

Abstract

We present a complete mathematical analysis of the dynamics encoded in a single discrete step of the Spiral of Theodorus, where one step corresponds to one unit of Planck time. The step operator $\mathcal{S}_n = 1 + i/\sqrt{n}$ is the irreducible building block of a lattice model in which the universe grows by appending one Planck unit of time, length, and mass per step. Each step is a coupled *growth-rotation* event: the real component $\mathrm{Re}(\mathcal{S}_n) = 1$ is the coordinate-defining contribution that advances the squared radius by exactly one integer, while the imaginary component $\mathrm{Im}(\mathcal{S}_n) = 1/\sqrt{n}$ is the phase-generating contribution that rotates the state vector without altering its radial projection. We derive five results from this operator alone: (i) the exact modulus identity $|z_n|^2 = n$; (ii) the action per step converges to exactly 1, providing a geometric origin for the quantisation of action ($\hbar$); (iii) the τ-loop within the step, parametrised by $\tau \in [1, e]$, which forces the emergence of Euler's number e as the loop boundary, π as the boundary phase, and the expansion eigenvalue $\Omega = \sqrt{\pi^e\, e^{1-e}} \approx 2.0071$ as the amplitude deposited at each collapse; (iv) the Planck momentum invariant, the fixed exchange rate between the coordinate and phase domains; (v) the construction of π and e from convergent integer series whose terms are drawn directly from the expansion index n, so that α is the only externally given constant. We identify a computational trade-off intrinsic to the model: as the universe expands, the integer series for π and e require more terms per step (the universe is larger), while the phase increment per step shrinks (the rotation slows), suggesting that precision and expansion rate are geometrically balanced.

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1 Introduction

1.1 Motivation

Modern physics rests on two postulates whose origin it does not explain: (a) action is quantised in units of $\hbar$, and (b) the wavefunction is complex. Both are introduced as axioms and justified by agreement with experiment. The present article derives both from the internal geometry of a single discrete step on the Spiral of Theodorus in the complex plane. The Spiral of Theodorus is a classical construction: beginning from a unit segment, one appends at each step a unit vector perpendicular to the current radius, generating a sequence of right triangles whose hypotenuses spiral outward with radii $1, \sqrt{2}, \sqrt{3}, \ldots$. When this construction is expressed as a complex recursion $z_{n+1} = z_n(1 + i/\sqrt{n})$, the resulting lattice encodes, without further input, a natural quantisation of action, a dual coordinate-and-phase structure, and a continuous inner dynamics that determines the transcendental numbers e and π as emergent boundary conditions.

1.2 The Central Theme: Growth + Rotation

The step operator $\mathcal{S}_n = 1 + i/\sqrt{n}$ is simultaneously a radial growth and an angular rotation. Every Planck step is a *coupled growth-rotation event*. This duality is not imposed; it is algebraically unavoidable for any complex number with both real and imaginary parts. It is the single structural fact from which all results in this article follow, and it persists at every scale of the model—from the individual Planck step analysed here, through the expanding lattice of Part 2, to the phase-closure conditions of Part 3.

1.3 Scope of This Article

This article—Part 1 of a three-part series—is concerned exclusively with the mathematics of *one step*: what happens during a single Planck time. We define the step, compute its modulus and phase, explain the dual domains it encodes, show the continuous τ-loop inside it, and derive the natural boundary conditions that fix e, π, and Ω. Parts 2 and 3 will address the macroscopic accumulation of steps (the expanding universe, field equations, and CMB observables) and the phase-closure conditions that select stable mass states (the base-15 cascade and particles).

1.4 Notation and Conventions

Throughout, $n \in \mathbb{Z}^+$ denotes the Planck-step index. All quantities are expressed in Planck units ($c = G = \hbar = k_B = 1$) unless otherwise stated. The complex lattice state at step n is z_n; its polar decomposition is $z_n = R_n\, e^{i\Theta_n}$. We write $R = \sqrt{n}$ when treating n as a continuous variable. The fine-structure constant is $\alpha \approx 1/137.036$; $\alpha_{\text{inv}} := 1/\alpha$. The expansion eigenvalue is $\Omega \approx 2.0071$; its derivation is the subject of Section 8.

2 The Discrete Step Operator

2.1 Definition

The Spiral of Theodorus is generated by the recursion

$$z_{n+1} = z_n \cdot \mathcal{S}_n, \qquad \mathcal{S}_n = 1 + \frac{i}{\sqrt{n}}, \qquad z_1 = 1. \tag{1}$$

The operator $\mathcal{S}_n$ adjoins a unit vector perpendicular to $z_n/|z_n|$ at each step, which is the defining geometric rule of the Theodorus construction. To verify:

$$z_{n+1} = z_n + z_n \cdot \frac{i}{\sqrt{n}} = z_n + i\,\frac{z_n}{|z_n|} \tag{2}$$

where the last equality uses $|z_n| = \sqrt{n}$ (derived below). The added vector $i\,z_n/|z_n|$ is the unit vector in the direction 90 counterclockwise from z_n, confirming the perpendicularity condition.

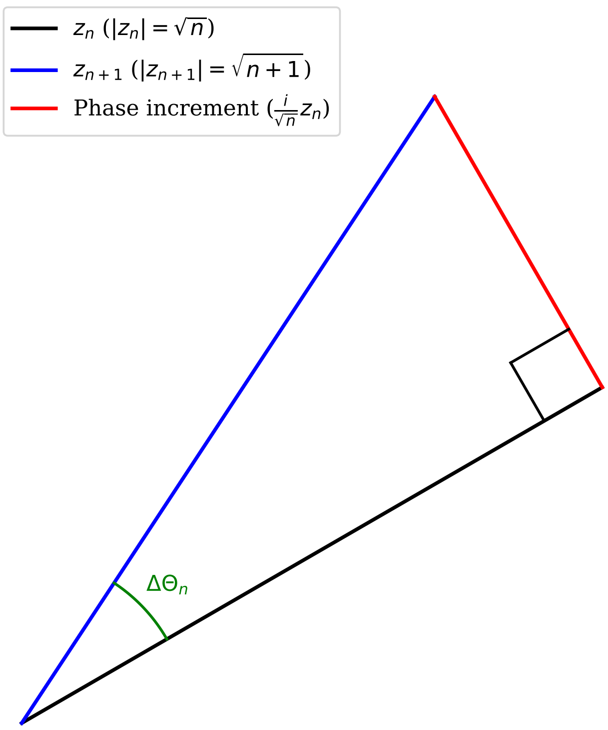

Figure 1: One step on the Theodorus spiral. The operator $\mathcal{S}_n$ defines a coupled growth-rotation event, adding exactly a perpendicular unit vector to the radius z_n.

2.2 Modulus and Argument of the Operator

The modulus of $\mathcal{S}_n$ is

$$|\mathcal{S}_n| = \left|1 + \frac{i}{\sqrt{n}}\right| = \sqrt{1 + \frac{1}{n}} = \sqrt{\frac{n+1}{n}}. \tag{3}$$

The argument of $\mathcal{S}_n$ is

$$\arg(\mathcal{S}_n) = \arctan\left(\frac{1}{\sqrt{n}}\right). \tag{4}$$

These are the only two properties of $\mathcal{S}_n$ that enter the analysis: at each step it stretches the radius by $\sqrt{(n+1)/n}$ and rotates the phase by $\arctan(1/\sqrt{n})$. This is the growth-rotation event in its most compact form.

3 Exact Results for a Single Step

3.1 The Modulus Identity: $|z_n|^2 = n$

Theorem. For all $n \in \mathbb{Z}^+$, $|z_n|^2 = n$.

Proof. By induction. Base case: $z_1 = 1$, so $|z_1|^2 = 1$. Inductive step: suppose $|z_n|^2 = n$. Then

$$|z_{n+1}|^2 = |z_n|^2 \cdot |\mathcal{S}_n|^2 = n \cdot \frac{n+1}{n} = n+1. \tag{5}$$

□ Alternatively, the modulus telescopes as a product:

$$|z_n| = \prod_{k=1}^{n-1} |\mathcal{S}_k| = \prod_{k=1}^{n-1} \sqrt{\frac{k+1}{k}} = \sqrt{\prod_{k=1}^{n-1} \frac{k+1}{k}} = \sqrt{\frac{2}{1} \cdot \frac{3}{2} \cdots \frac{n}{n-1}} = \sqrt{n}. \tag{6}$$

The product telescopes exactly because each numerator cancels the previous denominator: $\prod_{k=1}^{n-1}(k+1)/k = n/1 = n$. Writing $R_n := |z_n|$, we have the exact identity

$$\boxed{R_n^2 = n \qquad \text{(exact for all } n \in \mathbb{Z}^+\text{).}} \tag{7}$$

This is the defining property of what we will call the **coordinate domain**: the integer n is simultaneously the step index and the squared amplitude of the lattice state. The growth component of the step operator is responsible: each application of $\mathcal{S}_n$ advances $|z|^2$ by exactly 1.

3.2 The Phase Increment

The phase accumulated after n steps is

$$\Theta_n = \sum_{k=1}^{n-1} \arg(\mathcal{S}_k) = \sum_{k=1}^{n-1} \arctan\left(\frac{1}{\sqrt{k}}\right). \tag{8}$$

The phase increment contributed by a *single* step at position n is

$$\Delta\Theta_n := \Theta_{n+1} - \Theta_n = \arctan\left(\frac{1}{\sqrt{n}}\right). \tag{9}$$

For large n, $\arctan(1/\sqrt{n}) \approx 1/\sqrt{n}$, so the accumulated phase grows as

$$\Theta_n \approx \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \approx 2\sqrt{n} \quad (n \to \infty), \tag{10}$$

where the asymptotic form is Hlawka's theorem on the Spiral of Theodorus [5]. The phase grows as $\sqrt{n}$, not as n; it belongs to a different scaling class from the squared amplitude $R_n^2 = n$. This difference is the origin of domain duality (Section 4).

3.3 What One Step Produces: Summary

At step n, the operator $\mathcal{S}_n$ produces exactly one coupled growth-rotation event:

Quantity	**Exact value**	**Asymptotic form**
Growth: $\lvert z_{n+1}\rvert^2 - \lvert z_n\rvert^2$	1 (exact)	1
Radius $R_n = \lvert z_n\rvert$	$\sqrt{n}$	$\sqrt{n}$
Rotation: $\Delta\Theta_n$	$\arctan(1/\sqrt{n})$	$1/\sqrt{n}$
Accumulated phase Θ_n	$\sum_{k=1}^{n-1} \arctan(1/\sqrt{k})$	$2\sqrt{n}$

The growth is exactly 1 at every step (no approximation). The rotation is $\arctan(1/\sqrt{n})$, which decreases as the spiral expands.

4 Domain Duality: Coordinate-Defining and Phase-Generating

The step operator $\mathcal{S}_n$ encodes two fundamentally different contributions within a single complex number.

4.1 The Orthogonal Decomposition

Separating the operator into its real and imaginary parts:

$$\mathcal{S}_n = \underbrace{1}_{\text{coordinate-defining}} + \underbrace{\frac{i}{\sqrt{n}}}_{\text{phase-generating}} . \tag{11}$$

Real component: coordinate-defining. The constant real part 1 advances $|z_n|^2$ by exactly 1 per step. After n steps the squared radius is the integer n; it has a definite numerical value and can be assigned a coordinate. Quantities that depend on n (or equivalently on R_n^2) belong to the **coordinate (integer) domain**: mass, volume, step count, macroscopic time. They scale linearly with n. **Imaginary component: phase-generating.** The term $i/\sqrt{n}$ is a purely imaginary rotation of magnitude $1/\sqrt{n}$. It rotates the state vector z_n in the complex plane without contributing to the radial coordinate—any purely imaginary multiplication is orthogonal to the current direction. There is no real projection: the imaginary axis generates angular displacement, not radial position. Quantities that depend on the accumulated phase $\Theta_n \approx 2\sqrt{n}$ belong to the **phase ($\sqrt{\text{integer}}$) domain**: wavelength, oscillation frequency, temperature. They scale as $\sqrt{n}$.

4.2 Why the Phase Component Cannot Define a Coordinate

A reviewer may ask: why does the imaginary component correspond to a physical oscillation rather than a second spatial coordinate? The answer is structural. A purely imaginary multiplication $z \mapsto z + iz/|z|$ is a rotation: it is orthogonal to the current radial direction by construction. The result of a rotation is an angular displacement, not a position. To define a position requires a definite real value; but $e^{i\theta}$ is a unit-modulus complex number that sweeps through *all* real projections ($\cos\theta$ ranges over $[-1, 1]$) as θ varies. It carries phase information, not coordinate information. This is not a physical postulate about waves; it is a mathematical property of the imaginary unit: i generates rotations, and rotations do not define points.

4.3 The Two Scaling Laws

The separation into coordinate and phase domains produces two distinct scaling laws from the same lattice:

$$\text{Coordinate (integer) domain:} \quad R_n^2 = n, \quad \text{scales as } t_{\text{age}}, \tag{12}$$

$$\text{Phase } (\sqrt{\text{integer}}) \text{ domain:} \quad \Theta_n \approx 2\sqrt{n}, \quad \text{scales as } \sqrt{t_{\text{age}}}. \tag{13}$$

The ratio of phase to radius converges to a fixed value:

$$\frac{\Theta_n}{R_n} \longrightarrow 2 \qquad (n \to \infty). \tag{14}$$

This asymptotic slope of 2 is an intrinsic property of the Theodorus spiral, not a normalisation choice. It is the geometric origin of the factor of 2 that appears in the electromagnetic bridge coupling $\beta = \sqrt{2\alpha}$ (to be derived in Part 3).

4.4 Growth–Rotation Coupling

The two contributions are not independent; they are locked together by the structure of $\mathcal{S}_n$. At each step:

- The growth (real part) is exactly 1, independent of n.
- The rotation (imaginary part) is $1/\sqrt{n}$, *decreasing* as the lattice expands.
- Their ratio is $1/\sqrt{n}$: the rotation per unit growth shrinks as n increases.

This coupling is the mechanism by which a discrete, expanding lattice produces smooth, convergent geometry. At early times (n small), the rotation per step is large and the spiral winds tightly; at late times (n large), the rotation per step is small and the spiral unwinds slowly. The universe "decelerates" rotationally while maintaining constant radial growth.

4.5 Correspondence with the Physical States

In the broader framework developed in earlier articles of this series [1], the two components of the step operator correspond to two physical states:

State	**Contribution**	**Domain**	**Duration**
Point-state (coordinate-defining, mass)	Real: 1	Integer (n)	1 Planck time
Wave-state (phase-generating, radiation)	Imaginary: $i/\sqrt{n}$	$\sqrt{\text{Integer}}$	m_P/m Planck times

The point-state is the moment when the lattice has a definite coordinate ($|z|^2 = n$, an integer). The wave-state is the interval during which the lattice is undergoing phase rotation with no fixed radial projection. This identification is not an additional assumption; it is the operator (1) read off component by component.

5 Geometric Quantisation: Action → 1 Per Step

5.1 Definition of the Step Action

Having established that each step produces a coupled growth-rotation event, we now compute the *action* of a single step. Define the action per Planck step as the product of the growth amplitude and the rotation increment:

$$S_n := R_n \cdot \Delta\Theta_n = \sqrt{n} \cdot \arctan\left(\frac{1}{\sqrt{n}}\right). \tag{15}$$

This definition identifies R_n (the coordinate) and $\Delta\Theta_n$ (the phase increment) as conjugate variables: R_n is the generalised coordinate and $\Delta\Theta_n$ is the conjugate momentum.

5.2 Derivation of the Asymptotic Limit

Using the Taylor expansion $\arctan x = x - x^3/3 + x^5/5 - \cdots$ with $x = 1/\sqrt{n}$:

$$\begin{aligned} S_n &= \sqrt{n}\left(\frac{1}{\sqrt{n}} - \frac{1}{3n^{3/2}} + \frac{1}{5n^{5/2}} - \cdots\right) \\ &= 1 - \frac{1}{3n} + \frac{1}{5n^2} - \cdots \end{aligned} \tag{16}$$

Equivalently, by L'Hôpital's theorem applied to $\lim_{x\to 0}(\arctan x)/x = 1$:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} \cdot \frac{1}{\sqrt{n}} = 1. \tag{17}$$

Therefore:

$$\boxed{\lim_{n\to\infty} S_n = 1.} \tag{18}$$

Every Planck step contributes *exactly one unit of action* in the asymptotic limit.

5.3 The Convergence Rate

The leading correction is:

$$S_n = 1 - \frac{1}{3n} + O(n^{-2}). \tag{19}$$

Numerically:

n	S_n	Deficit $1 - S_n$
1	0.78540	0.215
10	0.98333	0.017
10^2	0.99833	1.7×10^{-3}
10^3	0.99983	1.7×10^{-4}
10^4	0.99998	1.7×10^{-5}
10^6	0.999998	1.7×10^{-7}
10	0.96853	0.031
10^2	0.99669	3.3×10^{-3}
10^3	0.99967	3.3×10^{-4}
10^4	0.99997	3.3×10^{-5}
10^6	$1 - 3.3 \times 10^{-7}$	3.3×10^{-7}
10^{61} (present universe)	$1 - O(10^{-62})$	$\sim 10^{-62}$

5.4 Physical Interpretation

The quantum of action $\hbar$ is not imposed on the lattice; it is the limiting value of the spiral's own step invariant S_n. The product $R_n \cdot \Delta\Theta_n$ is the *area swept* by the state vector during one step—the discrete analogue of the canonical commutation relation $[Q, P] = i\hbar$, with $Q = R_n$ (coordinate) and $P = \Delta\Theta_n$ (phase/momentum) as the conjugate pair. The first-order correction $S_n \approx 1 - 1/(3n)$ has a physical interpretation: the action deficit at early times (n small) is the geometric cost of discreteness. Only as the lattice matures ($n \to \infty$) does the action per step reach exact unity. The quantisation of action is not a primordial axiom but an asymptotic property of the expanding spiral.

6 The τ-Loop: Continuous Dynamics Within One Step

The preceding sections treated the step operator $\mathcal{S}_n$ as an instantaneous transformation from state n to state $n+1$. We now resolve the *internal structure* of this transformation: the continuous dynamics that occur within a single Planck step.

6.1 Physical Motivation

Between consecutive coordinate-defining moments (each producing $|z|^2 = n$, then $|z|^2 = n + 1$), the lattice is in a continuous phase-generating state. The amplitude must sweep from its value at the start of the step to its value at the end. This continuous sweep is the τ-loop. The τ-loop is the innermost of three nested oscillation levels in the full model (the other two—the electron cycle and the cosmological cycle—are developed in Parts 2 and 3). Each level shares the same architecture: a continuous phase-generating (wave) interval followed by a discrete coordinate-defining (point) moment.

6.2 The Wave-State Amplitude Function

The radial amplitude of the lattice *during the phase-generating interval* is described by the wave-state amplitude function:

$$A(\tau) = \Omega^{\ln \tau + 1}, \qquad \tau \in [1, e], \tag{20}$$

where Ω is the expansion eigenvalue (derived in Section 8). Within the framework of the Ω-exponent cascade (developed fully in Part 3), each physical quantity carries a characteristic power of Ω called its Ω-*exponent* k. The function $A(\tau) = \Omega^{\ln \tau + 1}$ sweeps the Ω-exponent continuously from $k = 1$ to $k = 2$ as τ advances from 1 to e:

$$A(1) = \Omega^{\ln 1 + 1} = \Omega^{0+1} = \Omega^1 = \Omega \qquad \text{(phase-domain bridge, } k = 1), \tag{21}$$

$$A(e) = \Omega^{\ln e + 1} = \Omega^{1+1} = \Omega^2 \qquad \text{(coordinate domain, } k = 2). \tag{22}$$

The form $A(\tau) = \Omega^{\ln \tau + 1}$ is the unique monotone function with these boundary conditions: the logarithm converts the multiplicative range $[1, e]$ into the additive range $[0, 1]$, and the $+1$ sets the starting level at $k = 1$ rather than $k = 0$. The Ω-exponent $k = 1$ corresponds to $\sqrt{\text{Planck momentum}}$, the bridge object between the coordinate and phase domains. The Ω-exponent $k = 2$ corresponds to the coordinate-domain objects (velocity, length). The τ-loop thus sweeps *from* the phase domain bridge *to* the coordinate domain within each step—a continuous transition from phase-generating to coordinate-defining.

6.3 The Winding Number $\psi_0 = 1$

The winding number ψ_0 of the Planck step is defined as the number of Ω-exponent levels traversed during one step:

$$\psi_0 := \Delta k = \frac{\ln(A(e)/A(1))}{\ln \Omega} = \frac{\ln(\Omega^2/\Omega)}{\ln \Omega} = \frac{\ln \Omega}{\ln \Omega} = 1 \quad \text{exactly.} \tag{23}$$

This result is independent of the numerical value of Ω: the Planck step has winding number 1 by the geometry of the τ-loop alone. This is the foundational quantisation condition of the model.

7 Emergence of e, π, and Ω from $\psi_0 = 1$

The winding condition $\psi_0 = 1$ is a single equation, but it determines three quantities: the τ-loop boundary (Euler's number e), the boundary phase (π), and the expansion eigenvalue (Ω). We derive each in turn.

7.1 Origin of e: The Loop Boundary

The winding condition $\psi_0 = 1$ requires exactly one unit of logarithmic action within the τ-loop. Since the wave-state amplitude is $A(\tau) = \Omega^{\ln \tau + 1}$, the Ω-exponent at the loop boundary is $\ln \tau_{\text{final}} + 1$, and the change in Ω-exponent is:

$$\Delta k = (\ln \tau_{\text{final}} + 1) - (\ln 1 + 1) = \ln \tau_{\text{final}} = 1. \tag{24}$$

The unique solution is

$$\boxed{\tau_{\text{final}} = e^1 = e.} \tag{25}$$

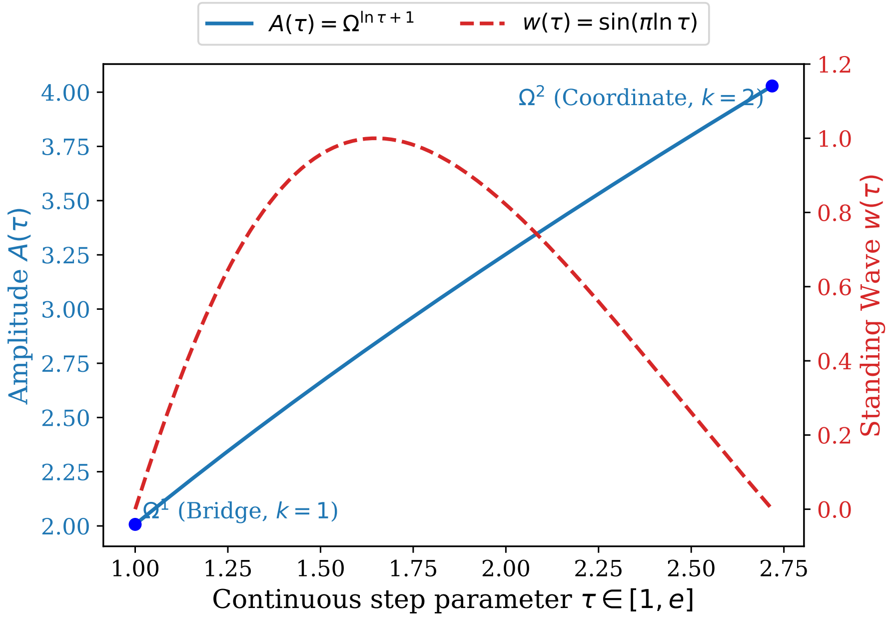

Figure 2: The τ-loop connecting the phase domain ($k = 1$) to the coordinate domain ($k = 2$). The wave amplitude $A(\tau)$ sweeps from Ω^1 to Ω^2, bounding the half-cycle standing wave $w(\tau)$.

Euler's number e is not chosen as a parameter of the model. It is the *unique real number whose natural logarithm equals 1*—the unique boundary at which the τ-loop traverses exactly one Ω-exponent level. Every property of e that appears in the model (the natural exponential, the compound interest limit $\lim(1 + 1/n)^n$, the base of natural logarithms) is a consequence of this single equation.

7.2 Origin of π: The Boundary Phase

Within the τ-loop, the phase-generating oscillation is described by a standing wave in the logarithmic coordinate $u = \ln\tau$:

$$w(\tau) = \sin(\pi \ln\tau). \tag{26}$$

This is the simplest oscillation that vanishes at both endpoints of the interval $u \in [0, 1]$ (i.e. $\tau \in [1, e]$):

$$w(1) = \sin(\pi \cdot 0) = 0 \qquad \text{(node at start)}, \tag{27}$$

$$w(e) = \sin(\pi \cdot 1) = 0 \qquad \text{(node at end)}. \tag{28}$$

The wave starts and ends at a node, completing exactly one half-cycle. The phase accumulated over the full τ-loop is:

$$\phi_{\text{boundary}} = \pi \ln e = \pi \cdot 1 = \pi. \tag{29}$$

In the MLTA algebra, the Planck time unit is $T = \pi$, which counts this half-cycle. The value π is not chosen; it is the *phase accumulated during exactly one unit of logarithmic action*. It is the unique eigenvalue of the boundary-value problem $\sin(\lambda \cdot 1) = 0$ with $\lambda > 0$ minimal, i.e. $\lambda = \pi$.

7.3 The Forcing Chain

The logical chain connecting the winding condition to the expansion eigenvalue is:

$$\psi_0 = 1 \implies \ln \tau_{\text{final}} = 1 \implies \tau_{\text{final}} = e \implies \phi = \pi \ln e = \pi \implies \Omega = \sqrt{F(e)}. \tag{30}$$

The final arrow is the subject of the next section.

8 The Capacity Functional and the Derivation of Ω

8.1 Two Irreducible Constants

The forcing chain of Section 7 has established that the τ-loop boundary yields e and the boundary phase yields π. A universe built from discrete integer steps (the Theodorus recursion $\mathcal{S}_n = 1 + i/\sqrt{n}$) and from circular geometry (closed orbits, standing waves) necessarily involves exactly these two transcendental constants:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \qquad \text{(natural growth, counting)}, \tag{31}$$

$$\pi = \frac{\text{circumference}}{\text{diameter}} \qquad \text{(circular geometry, periodicity)}. \tag{32}$$

No other transcendental constants arise inevitably from counting and geometry.

8.2 The Canonical Interpolation

The unique continuous function of the form $a^t b^{1-t}$ satisfying $f(0) = e$ and $f(1) = \pi$ is

$$f(t) = e^{1-t}\,\pi^t = e \cdot \left(\frac{\pi}{e}\right)^t. \tag{33}$$

This function smoothly maps the natural growth base e (at $t = 0$) to the circular geometry base π (at $t = 1$). It is strictly increasing ($f'(t) = f(t)\ln(\pi/e) > 0$ since $\pi > e$) and C^∞.

8.3 The Capacity Functional

We define the capacity functional as the product of the primitive unit e and the capacity ratio $(\pi/x)^x$:

$$F(x) = e \cdot \left(\frac{\pi}{x}\right)^x = \frac{e\,\pi^x}{x^x}, \tag{34}$$

which measures how much the spatial (π-based) accumulation exceeds the self-referential (x^x) accumulation at interaction parameter x. The choice of x^x in the denominator is deliberate: x^x is the self-consistent measure of exponential growth at rate x, the rate produced by the lattice itself.

8.4 Two Competing Constraints

$F(x)$ is governed by two independent constraints that the lattice must simultaneously satisfy: **Constraint 1: Geometric maximum.** Setting $d(\ln F)/dx = 0$:

$$\frac{d}{dx}\ln F(x) = \frac{d}{dx}\big[1 + x\ln\pi - x\ln x\big]$$
$$= \ln\pi - \ln x - 1 = \ln\Big(\frac{\pi}{x}\Big) - 1 = 0. \tag{35}$$

This yields
$$x_{\text{geom}} = \frac{\pi}{e} \approx 1.156. \tag{36}$$

This is the purely spatial constraint: the value of x that maximises the capacity of the π-based geometry relative to the self-referential cost. **Constraint 2: Self-referential fixed point.** The continuous limit of the spiral (the logarithmic sum (8)) is expressed in natural logarithms. For the lattice to remain self-consistent—to compute its own constants rather than assume an external base—the interaction parameter must equal the base of the natural logarithm:

$$\ln x = 1 \quad\Longrightarrow\quad x_{\text{nat}} = e. \tag{37}$$

This is the same condition that forces $\tau_{\text{final}} = e$ in the τ-loop. **The incommensurability.** The two constraints yield different values:

$$x_{\text{geom}} = \pi/e \approx 1.156 \quad\neq\quad x_{\text{nat}} = e \approx 2.718.$$

The lattice cannot satisfy both simultaneously. This incommensurability is the geometric strain: the system is locked between a spatial preference (maximise π-capacity) and a self-referential preference (use base e) with no common solution.

8.5 Resolution: Evaluation at the Natural Boundary

The $\psi_0 = 1$ winding condition has already selected $x = e$ as the natural boundary of the τ-loop (Section 7). Evaluating F at this boundary:

$$\Omega^2 = F(e) = e\cdot\Big(\frac{\pi}{e}\Big)^e = \frac{e\,\pi^e}{e^e} = \pi^e\,e^{1-e}. \tag{38}$$

Taking the positive square root (the lattice expands outward):

$$\boxed{\Omega = \sqrt{\pi^e\,e^{1-e}} \approx 2.0071349543\ldots} \tag{39}$$

8.6 The Role of e^{e-1}

The denominator factor $e^{e-1} = e^e/e$ has a natural interpretation: it is the self-referential cost x^x evaluated at $x = e$ (giving e^e), divided by the primitive unit e already carried by F. It measures the excess of the self-referential cost over the unit scale:

$$\Omega^2 = \frac{\pi^e}{e^{e-1}} = \frac{\text{geometric (π-based) accumulation}}{\text{self-referential excess cost}}. \tag{40}$$

No separate assertion is required; this is a direct consequence of evaluating $F(e)$.

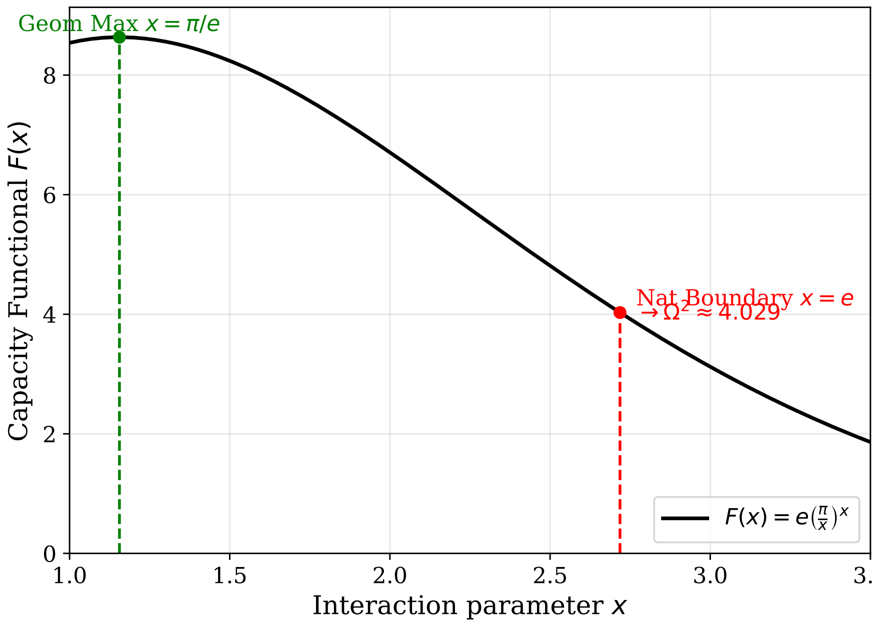

Figure 3: The capacity functional $F(x) = e(\pi/x)^x$. The lattice must evaluate $F(x)$ at the self-referential boundary $x = e$, producing the squared expansion eigenvalue $\Omega^2 \approx 4.029$.

8.7 The Square Root: Forced by Domain Duality

The Theodorus spiral has two distinct scaling domains (Section 4):

$$\text{Coordinate domain:} \quad \text{amplitude} \sim n \quad (\sim t),$$
$$\text{Phase domain:} \quad \text{radius} \sim \sqrt{n} \quad (\sim \sqrt{t}).$$

A bridge constant mediating these two domains must transform $t \leftrightarrow \sqrt{t}$. At the canonical time $t = e$:

$$\text{Coordinate unit:} \quad F(e) = \Omega^2 \quad (\text{scales as } t), \tag{41}$$
$$\text{Phase unit:} \quad \sqrt{F(e)} = \Omega \quad (\text{scales as } \sqrt{t}). \tag{42}$$

The bridge constant is therefore $\Omega = \sqrt{F(e)}$. Ω^2 enters wherever a coordinate-domain quantity is needed; Ω enters wherever a phase-domain quantity is needed. This is confirmed by the MLTA unit assignments: $V = 2\pi\Omega^2$ and $L = 2\pi^2\Omega^2$ (coordinate domain, Ω^2), while $\sqrt{p_P} = \Omega$ is the first-order bridge (Ω^1).

8.8 Summary of the Ω Derivation

Ω is not a tuning parameter. It is the unique value of the capacity functional $F(x) = e \cdot (\pi/x)^x$ at the natural e-fold boundary $x = e$, which is itself forced by the winding condition $\psi_0 = 1$. The complete derivation chain is:

$$\psi_0 = 1 \;\to\; \tau_{\text{final}} = e \;\to\; F(e) = \pi^e \, e^{1-e} \;\to\; \Omega = \sqrt{\pi^e \, e^{1-e}}.$$

Given the spiral step operator (1), Ω could not take any other value.

9 The Emergence of π and e from Integer Series

A critical feature of the model is that the transcendental constants π and e are not primordial: they do not exist at the birth of the universe. They emerge asymptotically as the lattice matures, constructed term by term from integer arithmetic—the only arithmetic available to a discrete counting process.

9.1 The Series

The model constructs π and e from the following convergent series, whose terms are drawn directly from the expansion index n: π **via the Nilakantha series:**

$$\pi_n = 3 + \sum_{k=1}^{n} \frac{(-1)^{k+1} \cdot 4}{(2k)(2k+1)(2k+2)}. \tag{43}$$

Every term involves only the integers $2k$, $2k+1$, $2k+2$ and the constant 4. The series converges as $|\pi_n - \pi| \sim 1/n^2$. e **via the factorial series:**

$$e_n = \sum_{k=0}^{n} \frac{1}{k!}. \tag{44}$$

Every term involves only the integer factorial $k!$. The series converges as $|e_n - e| \sim 1/(n+1)!$, which is superexponentially fast.

9.2 Self-Bootstrapping of Ω

Since $\Omega = \sqrt{\pi^e \, e^{1-e}}$, the expansion eigenvalue at step n is

$$\Omega_n = \sqrt{\pi_n^{e_n} \, e_n^{1-e_n}}, \tag{45}$$

a convergent sequence whose limit is the true Ω. The convergence is rapid:

n	π_n	e_n	Ω_n
1	$3.1\overline{6}$	2.0000	2.0298
5	3.14185	2.71667	2.00735
10	3.141593	2.7182818	2.007136
25	3.141592653	2.71828182844	2.00713495
1	3.16667	2.00000	2.23917
5	3.14271	2.71667	2.00890
10	3.14141	2.71828	2.00697
25	3.14161	2.71828	2.00715

At $n = 25$ the error in Ω_n is 0.0006% of the true value. The reference programme [4] demonstrates this self-bootstrapping numerically.

9.3 The Expanding Lattice as a FOR Loop

The spiral does not exist as a fixed geometric object. It grows by one Planck step at each unit of cosmic time. The construction is equivalent to:

$$\texttt{FOR}\ n = 1\ \texttt{TO}\ t_{\text{age}}\texttt{:}\quad \text{add}\ \{t_P,\, l_P,\, m_P\}\texttt{; compute}\ \pi_n,\, e_n,\, \Omega_n\texttt{; NEXT} \tag{46}$$

At each step, π_n and e_n are updated using one additional term of their respective series. The constant Ω_n is therefore not an axiom of the model but a *convergent output* of the expanding spiral.

9.4 α as the Only External Input

The integers that build π_n and e_n are generated by the expansion itself: the index n, its products, and its factorials are available to any counting process. No external mathematical constant is required to construct them. The sole empirical input to the full model is the fine-structure constant $\alpha \approx 1/137.036$. It does not appear in the spiral geometry (the step operator $\mathcal{S}_n$ contains no α), in the capacity functional ($F(x)$ contains no α), or in any result of this article. α enters only at the Compton scale, when the spiral has accumulated enough steps for the electromagnetic channel to open (at step $n_{\text{EM}} = \alpha_{\text{inv}}/2 \approx 68.5$). This is the subject of Part 3.

10 The Momentum Invariant

A key structural invariant of the single Planck step is the product of growth amplitude and phase rate, which we now derive.

10.1 Derivation

From the asymptotic forms established in Section 3, the amplitude at time t is $\sqrt{t}$ (coordinate domain) and the phase is $2\sqrt{t}$ (phase domain). The momentum invariant is the product of amplitude and the time-derivative of the phase:

$$\text{Amplitude} \times \frac{d}{dt}(\text{Phase}) = \sqrt{t} \times \frac{d}{dt}\Big(2\sqrt{t}\Big) = \sqrt{t} \times \frac{1}{\sqrt{t}} = 1. \tag{47}$$

This product locks to unity for all t. Planck momentum in this framework is a discrete, scale-invariant property of the lattice itself, not a continuous kinematic quantity.

10.2 Physical Interpretation

Equation (47) states that the *coordinate-domain amplitude* and the *phase-domain rate* are exact reciprocals at every step. As the spiral expands ($\sqrt{t}$ grows), the phase rate ($1/\sqrt{t}$) decreases by exactly the same factor. Growth and rotation are locked in a unit-product relationship: the lattice cannot grow without slowing its rotation, and cannot rotate faster without shrinking.

10.3 The Domain Bridge

Within the Ω-exponent scaffold (Part 3), $\sqrt{\text{Planck momentum}}$ carries Ω-exponent $k = 1$:

$$\sqrt{p_P} = \Omega\, r^2, \tag{48}$$

where r is the Planck radius unit. The assignment $k = 1$ places $\sqrt{p_P}$ in the phase domain, confirming that momentum is not a pure coordinate-domain quantity but the first-order bridge between the two domains. The full Planck momentum p_P scales as Ω^2, matching the coordinate-domain equilibrium amplitude.

10.4 The Domain Hierarchy

Equation (39) produces a natural three-tier hierarchy that maps onto the spiral's domain duality:

$$\Omega^2 = \pi^e\, e^{1-e} \approx 4.029 \qquad \text{(Coordinate/Mass domain: integer)} \tag{49}$$

$$\Omega \approx 2.007 \qquad \text{(Phase/}\sqrt{\text{Integer}}\text{ domain: bridge)} \tag{50}$$

$$\Omega^3 \approx 8.086 \qquad \text{(Charge/Volume domain)} \tag{51}$$

The transition from coordinate to charge involves multiplication by Ω—a step that corresponds (in Part 3) to a 120 rotation in the complex plane.

11 The Computational Trade-Off

The integer-series construction of π and e (Section 9) raises a structural question about the computational cost of each Planck step as the universe expands.

11.1 The Question

At step n, the Nilakantha series for π_n (equation 43) and the factorial series for e_n (equation 44) each require n terms to achieve their current precision. As the universe ages (n grows), more terms are needed—the computation per step increases. Simultaneously, the phase increment per step *decreases*:

$$\Delta\Theta_n = \arctan\left(\frac{1}{\sqrt{n}}\right) \approx \frac{1}{\sqrt{n}} \;\to\; 0 \qquad (n \to \infty). \tag{52}$$

The radial expansion per step also diminishes:

$$\Delta R_n = R_{n+1} - R_n = \sqrt{n+1} - \sqrt{n} \approx \frac{1}{2\sqrt{n}} \;\to\; 0. \tag{53}$$

Is there a trade-off? The greater computational work required (because the universe is larger and the series need more terms) is accompanied by a smaller geometric increment (the rotation and radial growth both shrink).

11.2 Quantifying the Trade-Off

We can make this precise. At step n:

Quantity	Scales as	Behaviour
Series terms for π_n (Nilakantha)	n	Increases
Series terms for e_n (factorial)	n	Increases
Phase increment $\Delta\Theta_n$	$1/\sqrt{n}$	Decreases
Radial increment ΔR_n	$1/(2\sqrt{n})$	Decreases
Action per step S_n	$1 - 1/(3n)$	Approaches 1
Precision of π_n (error)	$1/n^2$	Improves
Precision of e_n (error)	$1/n!$	Improves (fast)

The computational cost (series length) grows as n, while the geometric output per step (rotation, radial growth) shrinks as $1/\sqrt{n}$. The product is:

$$(\text{computational cost}) \times (\text{geometric output}) \sim n \times \frac{1}{\sqrt{n}} = \sqrt{n}. \tag{54}$$

This grows as $\sqrt{n}$, which is the *phase-domain* scaling. The computational trade-off is therefore not a cancellation but a domain transformation: the cost of maintaining precision grows in the coordinate domain ($\sim n$), while the output per step lives in the phase domain ($\sim 1/\sqrt{n}$), and their product returns the phase-domain radius $\sqrt{n}$.

11.3 Interpretation: Precision versus Expansion

There are two distinct ways the series might be evaluated at each step, corresponding to two different computational architectures: **Architecture A (incremental):** At step n, the series is extended by one term from step $n-1$. The cost per step is $O(1)$ (one additional term). The cumulative precision after n steps is $|\pi_n - \pi| \sim 1/n^2$. This corresponds to a universe that carries its computational history forward, using the previous step's value of π_{n-1} as the starting point. **Architecture B (recalculation):** At each step n, the full series is recomputed from $k = 1$ to $k = n$. The cost per step is $O(n)$. This corresponds to a universe that recomputes its constants from scratch at every step, using the current lattice size n as the number of available integer terms. Under Architecture B, there is a genuine trade-off: as the universe grows, each step requires more computation to achieve its current precision, but each step also contributes a smaller geometric increment. The simultaneous growth in series length and shrinkage in step size suggests a conservation principle: the total "computational work per unit of geometric output" may be bounded. Specifically, the precision of π_n improves as $1/n^2$ (Nilakantha convergence), while the geometric increment $\Delta\Theta_n$ shrinks as $1/\sqrt{n}$. The ratio

$$\frac{\text{precision improvement}}{\text{step size}} = \frac{1/n^2}{1/\sqrt{n}} = \frac{1}{n^{3/2}} \tag{55}$$

shrinks faster than either factor alone: the universe asymptotically requires less precision improvement per unit of geometric output, even as both the precision and the output individually change. The system is self-regulating.

11.4 The Self-Referential Moment at $n = \Omega^2$

A distinguished step occurs at $n = \Omega^2 \approx 4.029$. At this step the operator becomes

$$\mathcal{S}_{\Omega^2} = 1 + \frac{i}{\sqrt{\Omega^2}} = 1 + \frac{i}{\Omega}. \tag{56}$$

The phase-generating component is exactly $1/\Omega$—the reciprocal of the expansion eigenvalue. This is the unique self-referential moment of the lattice:

- The coordinate-domain amplitude is Ω.
- The phase-domain amplitude is $1/\Omega$.
- Their product is $\Omega \cdot (1/\Omega) = 1$: the lattice unit.

No other value of Ω preserves this unit product at the self-referential step. This is a consistency check on the derivation of Section 8: the Ω derived from the capacity functional is the same Ω that produces a unit product at $n = \Omega^2$.

12 Conclusion

Every Planck step is a coupled growth-rotation event. Starting from the step operator $\mathcal{S}_n = 1+i/\sqrt{n}$—a purely algebraic object for every finite n—we have derived the following results, all pertaining to the dynamics of a single step:

1. **The exact modulus identity $|z_n|^2 = n$.** The real (coordinate-defining) component of $\mathcal{S}_n$ advances the squared radius by exactly 1 per step. The result telescopes via $\prod(k+1)/k = n$ (Section 3.1).
2. **The asymptotic phase $\Theta_n \approx 2\sqrt{n}$.** The imaginary (phase-generating) component accumulates angular displacement scaling as $\sqrt{n}$, establishing a second domain with different scaling (Section 3.2).
3. **Domain duality: coordinate and phase.** The real part defines coordinates ($R_n^2 = n$, integer); the imaginary part generates phase ($\Theta_n \approx 2\sqrt{n}$, irrational). The imaginary component cannot define a coordinate because purely imaginary multiplication is a rotation, and rotations do not fix points (Section 4).
4. **Action per step $\to 1$.** The product $R_n \cdot \Delta\Theta_n \to 1$ provides a geometric origin for $\hbar$. The convergence $S_n = 1 - 1/(3n) + O(n^{-2})$ makes the quantisation an asymptotic property of the expanding spiral (Section 5).
5. **The τ-loop and $\psi_0 = 1$.** Within each step, a continuous τ-loop sweeps the Ω-exponent from $k = 1$ to $k = 2$. The winding number is exactly 1, independent of Ω (Section 6).
6. **e, π, and Ω are forced.** $\psi_0 = 1$ forces $\tau_{\text{final}} = e$; the standing wave forces $\phi = \pi$; the capacity functional at $x = e$ yields $\Omega = \sqrt{\pi^e\, e^{1-e}}$ (Sections 7–8).
7. **π and e from integers.** Both constants are constructed from convergent integer series (π_n via Nilakantha, e_n via factorials), making α the only external input (Section 9).

8. **The momentum invariant.** $\sqrt{t}\cdot d(2\sqrt{t})/dt = 1$: growth and rotation are locked in a unit product at every step (Section 10).

9. **A computational trade-off.** The series length grows as n while the geometric output shrinks as $1/\sqrt{n}$; their product returns the phase-domain radius $\sqrt{n}$, suggesting a self-regulating balance between precision and expansion (Section 11).

These results follow from the geometry of the Theodorus spiral and nothing else. The fine-structure constant α does not appear in this article because it plays no role at the single-step level; it enters only when steps are accumulated and the electromagnetic coupling scale is reached.

12.1 Outlook: Parts 2 and 3

- **Part 2: The expanding lattice.** The accumulation of $t_{\text{age}} \sim 10^{61}$ steps; the continuous-limit field equation $d\Psi/dR = (2i + 1/R)\Psi$ and its non-Hermitian Hamiltonian with complex Coulomb potential $V(R) = 4i/R$; the CMB temperature, Hubble constant, and cosmological predictions.

- **Part 3: Phase closure and particles.** The global step operator $\mathcal{S} = \Omega\, e^{i2\pi/3}$; the $\mathbb{Z}_3$ phase symmetry and $SU(3)$; the base-15 phase cascade with periods 3 (charge) and 5 (temperature); mass as phase-closure resonance at $k = \text{lcm}(3, 5) = 15$; the three oscillation levels (ψ_0, ψ_1, ψ_2).

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